Instructions:

1. Answer all problems.

2. There are 2 problems with a total of 40 points.

3. Upsampling and downsampling operations are defined mathematically in the sequence domain (see below). You do not have to derive the relationships in the frequency domain. For example, if you know the relationship between $X_u\left(e^{j\omega}\right)$ and $X\left(e^{j\omega}\right)$, then simply use it if it is pertinent to your answer.

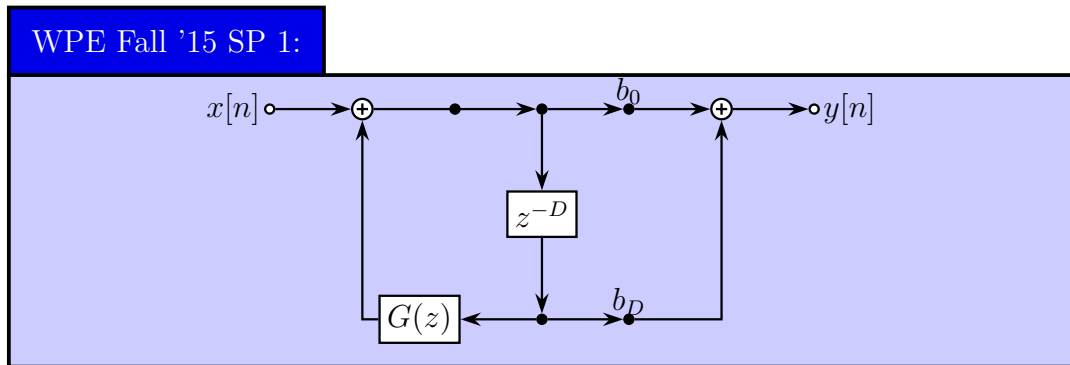4. Show work for partial credit.

Upsampling/Downsampling:

$x[n]$ → ↑ L → $x_u[n]$ (Upsampling)

$$x_u[n] = \begin{cases} x[n/L] & \text{for } n = 0, \pm L, \pm 2L, \ldots, \\ 0 & \text{otherwise.} \end{cases}$$

$x[n]$ → ↓ M → $x_d[n]$ (Downsampling)

$$x_d[n] = x[Mn]$$
$$= p[Mn]x[Mn].$$

Where $p[n]$ is a periodic unit pulse with period M.

1. Consider the following system:

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For parts (a) – (c), assume $b_0 = -a, b_D = 1$ and $G(z) = a$ with $0 < a < 1$

(a) Derive the difference equation for the output, $y[n]$.   [5 points]

    Solution. Define $w[n] = x[n] + aw[n - D]$, then

$$y[n] = b_0 w[n] + b_D w[n - D]$$
$$= -aw[n] + w[n - D]$$
$$= ay[n - D] - ax[n] + x[n - D] \quad (\star)$$

(b) Derive the relationship between $Y\left(e^{j\omega}\right)$ and $X\left(e^{j\omega}\right)$ and sketch the magnitude response of the filter. You will not need specific values for D and a, but you may assume $D = 8$ and $a = 0.75$.   [5 points]

    Solution. Take the DTFT of $(\star)$ to obtain

$$Y\left(e^{j\omega}\right) = \frac{-a + e^{-j\omega D}}{1 - ae^{-j\omega D}} X\left(e^{j\omega}\right) \Longrightarrow H\left(e^{j\omega}\right) = \frac{-a + e^{-j\omega D}}{1 - ae^{-j\omega D}}$$

    The magnitude response, $M(\omega)$ is given by

$$M(\omega) = \sqrt{\frac{(\cos(\omega D) - a)^2 + \sin^2(\omega D)}{(1 - a\cos(\omega D))^2 + a^2 \sin^2(\omega D)}} = 1, \forall \omega$$

    This is an allpass filter!

(c) Show that the unit pulse response[1] can be written in the form:

$$h[n] = c_1 \delta[n] + c_2 \left(a\delta[n - D] + a^2 \delta[n - 2D] + a^3 \delta[n - 3D] + \cdots\right)$$

    Explain why this filter is used to model reverberations in audio applications.
                                                                [5 points]

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[1]Some authors refer to this as the impulse response.

QUESTION 1(c) CONTINUES OVER THE PAGE

1(c) (Continued)

Solution. Use long division to obtain

$$H\left(z\right) = -\frac{1}{a} + \frac{1-a^2}{a}\frac{1}{1-az^{-D}}$$

$$= -a + \frac{1-a^2}{a}\left(az^{-D} + a^2 z^{-2D} + \cdots\right)$$

$$\implies$$

$$h[n] = -a\delta[n] + \frac{1-a^2}{a}\left(a\delta[n-D] + a^2\delta[n-2D] + \cdots\right)$$

For a given input sequence, $x[n]$, the output is

$$y[n] = -ax[n] + \frac{1-a^2}{a}\left(ax[n-D] + a^2 x[n-2D] + \cdots\right)$$

i.e. repeated and scaled replicas of the input. This usually results from reverberations of acoustic signals in concert halls.

(d) Now assume $b_0 = 1, b_D = 0$ and $G(z) = a(1+z^{-1}+z^{-2})$. Derive an expression for $h[n]$ and explain why the unit pulse response becomes denser for large values of n.   [5 points]

**Solution.** Evaluating the transfer function using the block diagram, we have

$$H\left(z\right) = \frac{1}{1 - z^{-D}G(z)}$$

$$= 1 + z^{-D}G(z) + z^{-2D}G^2(z) + z^{-3D}G^3(z) + \cdots$$

$$\implies \text{Giving the impulse response}$$

$$h[n] = \delta[n] + g[n-D] + \{g*g\}[n-2D] + \{g*g*g\}[n-3D] + \cdots$$

The impulse response is nonzero at $n = 0, D \le n \le D+2, 2D \le n \le D+4, \ldots, n \ge 15D$ due to the repeated convolution of $g[n]$ at each multiple of D. For example, for $D = 30$, $h[n]$ is initially sparse, but becomes dense for $n \ge 15D$.

2. Consider the two upsampling and interpolation systems shown in the figure:

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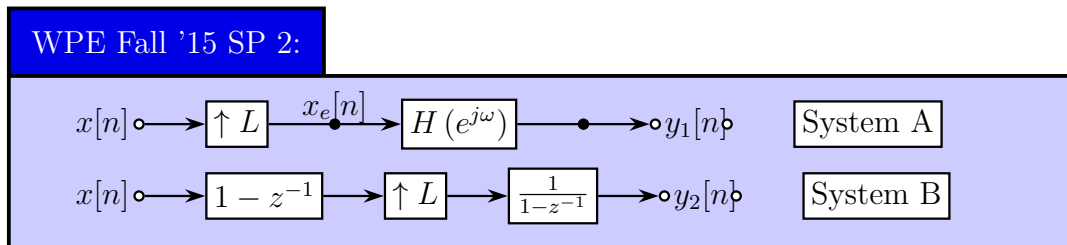
In System A, the frequency response of the filter is given by

$$H\left(e^{j\omega}\right) = \begin{cases} L, & |\omega| < \frac{\pi}{L} \\ 0, & \frac{\pi}{L} < |\omega| \le \pi. \end{cases}$$

Assume

$$x[n] = \frac{\sin(\omega_N n)}{\pi n}, \qquad \omega_N < \pi$$

as the input for both systems and answer the following questions.

(a) Sketch $|X_e\left(e^{j\omega}\right)|$ and $|Y_1\left(e^{j\omega}\right)|$. You may assume $L = 3$.   [5 points]
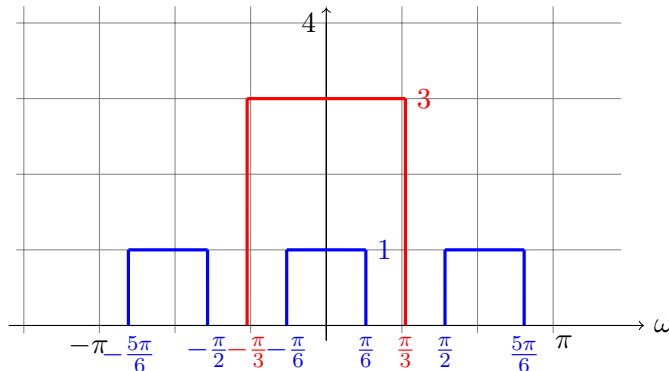
   Solution.

$$X\left(e^{j\omega}\right) = 1_{[-\omega_N, \omega_N]}$$
$$= \begin{cases} 1, & |\omega| < \omega_N \\ 0, & \omega_N < |\omega| \le \pi. \end{cases}$$

   After the expander

$$X_e\left(e^{j\omega}\right) = X\left(e^{j\omega L}\right)$$
$$= 1_{[-\omega_N/L, \omega_N/L]} \qquad \text{periodic with period} = 2\pi/L$$
$$\implies$$
$$Y_1\left(e^{j\omega}\right) = \begin{cases} 1, & |\omega| < \frac{\omega_N}{L} \\ 0, & \frac{\omega_N}{L} < |\omega| \le \pi. \end{cases}$$

   The situation is shown in the figure below for $\omega_N = \pi/2$ and $L = 3$. The ideal filter passes the image centered at $\omega = 0$ and completely rejects the two images centered at $\omega = \pm 2\pi/3$.

2(a) (Continued)

(b) Sketch $|Y_2(e^{j\omega})|$. You may assume $L = 3$. Comment on the nature of imperfection in signal interpolation.   [10 points]

Solution. In System B, the expander operates on $X(e^{j\omega})(1 - e^{-j\omega})$, producing $X(e^{j\omega L})(1 - e^{-j\omega L})$. It follows:

$$Y_1(e^{j\omega}) = \underbrace{X(e^{j\omega L})}_{Upsampler}\underbrace{\frac{1 - e^{-j\omega L}}{1 - e^{-j\omega}}}_{LPF} \implies |Y_1(e^{j\omega})| = |X(e^{j\omega L})|\frac{|\sin\omega L/2|}{\sin\omega/2}$$

This is an upsampler followed by a LPF. It is not an ideal interpolator due to imperfect suppresion of the spectral images created by the upsampler at multiples of $\frac{2\pi}{L}$.

The figure below shows the situation for $\omega_N = \pi/2$ and $L = 3$. Note that the filter does not completely reject the two images centered at $\pm 2\pi/3$. Furthermore, since the frequency response of the filter is not flat in the passband, it results in distortion of the desired signal in the band $\omega \in [-\pi/6, \pi/6]$.
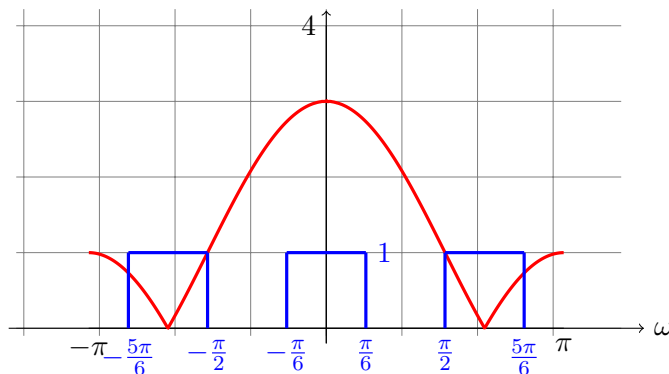
Note that the solution to Part (b) could be obtained by invoking the upsampling identity. Using this approach, the $1 - z^{-1}$ filter can be moved to the right of the expander and taking the form $1 - z^{-L}$. This would be great, but it is not necessary for the students to know this result in order to solve this problem.

QUESTION 2(b) CONTINUES OVER THE PAGE

2(b) (Continued)

(c) Comment on the merits and shortcomings of the two systems in terms of the feasibility and cost of implementation.   [5 points]

> **Solution.** The idea filter is not realizable and cannot be implemented without approximation allowing for some ripple in the passband and stop band with finite transition between the bands. The second filter is a poor approximation of the interpolation filter. However, it is very computationally efficient and can be implemented without multiplications. The performance of the filter can be improved by cascading more than one stage like the one shown in System B. This is the basis for the Cascaded Integrator Comb (CIC) filter which is widely used in practice, especially in FPGAs and VLSI signal processing.